

MAXIMAL ZERO SEQUENCES FOR FOCK SPACES

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ABSTRACT. A sequence Z in the complex plane \mathbb{C} is called a zero sequence for the Fock space F_α^p if there exists a function $f \in F_\alpha^p$, not identically zero, such that Z is the zero set of f , counting multiplicities. We show that there exist zero sequences Z for F_α^p with the following properties: (1) For any $a \in \mathbb{C}$ the sequence $Z \cup \{a\}$ is no longer a zero sequence for F_α^p ; (2) the space I_Z consisting of all functions in F_α^p that vanish on Z is one dimensional. These Z are naturally called maximal zero sequences for F_α^p .

1. INTRODUCTION

Let Ω be a domain in the complex plane \mathbb{C} and let X be a space of analytic functions on Ω . A sequence Z in Ω is called a zero sequence (or zero set) for X if there exists a function $f \in X$ such that f vanishes exactly on Z , counting multiplicities.

A classical example is the Hardy space H^p of the unit disk \mathbb{D} . In this case, $Z = \{z_n\}$ is a zero sequence for H^p if and only if

$$\sum_n (1 - |z_n|) < \infty,$$

which is called the Blaschke condition. Other examples that have been extensively studied in complex analysis include Bergman spaces, the Dirichlet space, and the disk algebra. See [1, 2, 4].

In all the examples mentioned above, if Z is a zero sequence for X , then

- (i) $Z \cup \{a_1, \dots, a_k\}$ remains a zero sequence for X , where a_1, \dots, a_k are arbitrary additional points from the underlying domain.
- (ii) The space I_Z of functions in X that vanish on Z (not necessarily exactly on Z) is always infinite dimensional.

In fact, it is easy to see that both (i) and (ii) hold whenever the space X is invariant under multiplication by polynomials. More generally, properties (i) and (ii) hold whenever X satisfies the following condition: for any point a in the underlying domain, there exists a function f_a such that f_a vanishes exactly at a and f_a is a pointwise multiplier of X . The underlying domain

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does not have to be bounded and polynomials do not need to be pointwise multipliers. In particular, Hardy spaces of the upper half-plane satisfy this condition.

The purpose of this note is to examine properties (i) and (ii) above in the case of Fock spaces. Thus for any $0 < \alpha < \infty$ and $0 < p < \infty$ we consider the Fock space F_α^p , consisting of entire functions f such that

$$\|f\|_{p,\alpha}^p = \frac{\alpha}{\pi} \int_{\mathbb{C}} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) < \infty.$$

When $p = \infty$, we define F_α^∞ to be the space of entire functions f such that

$$\|f\|_{\infty,\alpha} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

See [3, 9, 12] for more information about Fock spaces.

We will show that there exist zero sequences for F_α^p such that neither (i) nor (ii) holds. In fact, we will produce examples of zero sequences Z for F_α^p such that

- (i) For any $a \in \mathbb{C}$ the sequence $Z \cup \{a\}$ is no longer a zero sequence for F_α^p .
- (ii) $\dim(I_Z) = 1$.

These are clearly very extreme cases. Such Z will be called maximal zero sequences for F_α^p , because it cannot be expanded to another zero sequence for F_α^p . See [12] for other pathological properties of zero sequences for Fock spaces.

Note that $Z' = Z \cup \{a_1, \dots, a_k\}$ should not be looked at as a purely set-theoretic notation. It should be understood in the context of zero sequences for analytic functions and multiplicities of the zeros are part of the notation as well. For example, if $Z' = \{1, 1, 2\} \cup \{2, 2, 4\}$, then a function f vanishes on Z' when f has a zero of order 2 at $z = 1$, a zero of order 3 at $z = 2$, and a simple zero at $z = 4$. If we do not say that f vanishes *exactly* on Z' , then additional zeros are permitted.

2. WEIERSTRASS σ -FUNCTIONS

Our examples will be based on square lattices in the complex plane and the associated Weierstrass σ -functions. More specifically, for any $0 < \alpha < \infty$ we consider the square lattice

$$\Lambda_\alpha = \left\{ \omega_{mn} = \sqrt{\frac{\pi}{\alpha}} (m + in) : (m, n) \in \mathbb{Z}^2 \right\},$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ denotes the set of all integers. The Weierstrass σ -function associated to Λ_α is defined by

$$\sigma_\alpha(z) = z \prod_{(m,n) \neq (0,0)} \left[\left(1 - \frac{z}{\omega_{mn}} \right) \exp \left(\frac{z}{\omega_{mn}} + \frac{z^2}{2\omega_{mn}^2} \right) \right].$$

It is well known that σ_α is an entire function and its zeros are exactly the points in the lattice Λ_α .

The Weierstrass σ functions play an essential role in the study of Fock spaces. For example, these functions are used in [7, 8] to characterize interpolating and sampling sequences for Fock spaces. There are two properties of the Weierstrass σ -functions that will be critical to us here. We state them as the following two lemmas.

Lemma 1. *For any $\alpha > 0$ the function*

$$f(z) = |\sigma_\alpha(z)| e^{-\frac{\alpha}{2}|z|^2}$$

is doubly periodic with periods $\sqrt{\pi/\alpha}$ and $\sqrt{\pi/\alpha}i$.

Proof. See [10]. □

As a consequence of this result, we see that $\sigma_\alpha \in F_\alpha^\infty$, but $\sigma_\alpha \notin F_\alpha^p$ for any $0 < p < \infty$. In fact, if we let R_{mn} denote the square centered at ω_{mn} with horizontal and vertical side-lengths $\sqrt{\pi/\alpha}$, then by Lemma 1 and a change of variables,

$$\begin{aligned} \|\sigma_\alpha\|_{p,\alpha}^p &= \frac{\alpha}{\pi} \int_{\mathbb{C}} \left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \\ &= \frac{\alpha}{\pi} \sum_{m,n} \int_{R_{mn}} \left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \\ &= \frac{\alpha}{\pi} \sum_{m,n} \int_{R_{00}} \left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \\ &= \infty. \end{aligned}$$

It is well known that every function $f \in F_\alpha^p$ satisfies the following point-wise estimate

$$|f(z)| \leq \|f\|_{p,\alpha} e^{\frac{\alpha}{2}|z|^2}, \quad z \in \mathbb{C}. \quad (1)$$

See [3, 9, 12]. It follows from this and the definition of Fock spaces that $F_\alpha^p \subset F_\beta^q$ whenever $0 < \alpha < \beta < \infty$, where $0 < p \leq \infty$ and $0 < q \leq \infty$. Therefore, for $0 < \alpha_1 < \alpha < \alpha_2 < \infty$ and $0 < p \leq \infty$, we always have $\sigma_\alpha \in F_{\alpha_2}^p$ but $\sigma_\alpha \notin F_{\alpha_1}^p$.

Lemma 2. *If $0 < p < \infty$ and f is a function in F_α^p that vanishes on Λ_α , then f is identically zero.*

Proof. See [8]. □

Consequently, the square lattice Λ_α is a zero sequence for F_α^∞ but not a zero sequence for F_α^p when $0 < p < \infty$. Another consequence is the following.

Corollary 3. *Suppose $0 < \alpha < \beta < \infty$, $0 < p \leq \infty$, and $0 < q \leq \infty$. Then every zero sequence for F_α^p is a zero sequence for F_β^q , but there exists a zero sequence for F_β^q that is not a zero sequence for F_α^p .*

Proof. The first assertion follows from the embedding $F_\alpha^p \subset F_\beta^q$. To prove the second assertion, pick some $\gamma \in (\alpha, \beta)$. By Lemma 1, the function σ_γ belongs to F_β^q , so that Λ_γ is a zero sequence for F_β^q . If f is a function in $F_\alpha^p \subset F_\gamma^2$ that vanishes on Λ_γ , then by Lemma 2, f must be identically zero, so Λ_γ is not a zero sequence for F_α^p . □

The more interesting problem for us is when $\alpha = \beta$: do F_α^p and F_α^q have different zero sequences whenever $p \neq q$? Although we do not have a complete answer, it is easy to exhibit particular examples of such pairs that do not have the same zero sequences.

The simplest example is $Z = \Lambda_\alpha$, which is a zero sequence for F_α^∞ , but not a zero sequence for any F_α^p when $0 < p < \infty$. Similarly, $Z = \Lambda_\alpha - \{0\}$ is a zero sequence for F_α^p when $p > 2$, because the function $f(z) = \sigma_\alpha(z)/z$ belongs to F_α^p if and only if $p > 2$ (see Lemma 1 and calculations below). However, this Z is not a zero sequence for F_α^2 . To see this, suppose f is a function in F_α^2 such that f vanishes on Z . By Weierstrass factorization, we have $f(z) = [\sigma_\alpha(z)/z]g(z)$ for some entire function g . Let $\Omega = \mathbb{C} - R_{00}$ and

$$I(f) = \int_{\Omega} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^2 dA(z).$$

Then by the double periodicity of the function $\sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2}$,

$$\begin{aligned} I(f) &= \sum_{(m,n) \neq (0,0)} \int_{R_{mn}} \left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^2 \left| \frac{g(z)}{z} \right|^2 dA(z) \\ &= \sum_{(m,n) \neq (0,0)} \int_{R_{00}} \left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^2 \left| \frac{g(z + \omega_{mn})}{z + \omega_{mn}} \right|^2 dA(z). \end{aligned}$$

Let D denote the disk $|z| < \sqrt{\pi/\alpha}/100$ and choose a positive constant C_1 such that

$$|\sigma_\alpha(z)| e^{-\frac{\alpha}{2}|z|^2} > C_1, \quad z \in R_{00} - D.$$

We then have

$$I(f) \geq C_1 \sum_{(m,n) \neq (0,0)} \int_{R_{00}-D} \left| \frac{g(z + \omega_{mn})}{z + \omega_{mn}} \right|^2 dA(z).$$

When $(m, n) \neq (0, 0)$, the function $h(z) = g(z + \omega_{mn})/(z + \omega_{mn})$ is analytic on R_{00} . It follows easily from the subharmonicity of $|h(z)|^2$ that there is another positive constant C_2 , independent of (m, n) , such that

$$I(f) \geq C_2 \sum_{(m,n) \neq (0,0)} \int_{R_{00}} \left| \frac{g(z + \omega_{mn})}{z + \omega_{mn}} \right|^2 dA(z) = C_2 \int_{\Omega} \left| \frac{g(z)}{z} \right|^2 dA(z).$$

It is easy (using polar coordinates, for example) to show that

$$\int_{\Omega} \left| \frac{g(z)}{z} \right|^2 dA(z) < \infty$$

if and only if g is identically zero (here the exponent 2 is critical). Therefore, $f \in F_{\alpha}^2$ implies that f is identically zero. In other words, the sequence Z cannot possibly be a zero set for F_{α}^2 .

The above argument actually shows that $Z = \Lambda_{\alpha} - \{0\}$ is a uniqueness set for F_{α}^2 . Recall that a set Z in \mathbb{C} is called a uniqueness set (or set of uniqueness) for F_{α}^p if every function in F_{α}^p that vanishes on Z must be identically zero. It is also easy to see that the arguments above still work if the point 0 is replaced by any other point in Λ_{α} .

On the other hand, if Z is the resulting sequence when two points a and b are removed from Λ_{α} , then the function

$$f(z) = \frac{\sigma_{\alpha}(z)}{(z-a)(z-b)}$$

belongs to F_{α}^2 and has Z as its zero sequence. Therefore, Z is a zero set for F_{α}^2 . Consequently, it is possible to go from a uniqueness set to a zero set by removing just one point. Equivalently, it is possible to add just a single point to a zero set of F_{α}^2 so that the resulting sequence becomes a uniqueness set for F_{α}^2 . This shows how unstable the zero sets for Fock spaces are! See [5, 6] for applications of this rigidity in quantum physics.

We also observe that for any positive integer N with $Np > 2$, if Z is an F_{α}^q zero set, where $0 < p < q \leq \infty$, and if N points are removed from Z , then the remaining sequence becomes an F_{α}^p zero set. In fact, if Z is the zero sequence of a function $f \in F_{\alpha}^q$, not identically zero, and $Z' = Z - \{z_1, \dots, z_N\}$, then Z' is the zero sequence of the function

$$g(z) = \frac{f(z)}{(z-z_1) \cdots (z-z_N)},$$

which is easily seen to be in F_α^p . In fact, if $R > \max(|z_1|, \dots, |z_N|)$, then it follows from the pointwise estimate (1) that there exists a positive constant C such that

$$\int_{|z|>R} |g(z)e^{-\frac{\alpha}{2}|z|^2}|^p dA(z) \leq C \int_{|z|>R} \frac{dA(z)}{|z - z_1|^p \cdots |z - z_N|^p} < \infty.$$

Therefore, zero sets for F_α^p and F_α^q may be different, but they are not that much different! In other words, the difference is only in a finite number of points.

3. MAXIMAL ZERO SETS FOR F_α^p

Let Z be a zero sequence for F_α^p and let I_Z denote the set of functions f in F_α^p such that f vanishes on Z (but not necessarily exactly on Z). In the classical theories of Hardy and Bergman spaces, the space I_Z is always infinite dimensional. This is no longer true for Fock spaces.

Theorem 4. *Let k be any positive integer or ∞ . Then there exists a zero sequence Z for F_α^p such that $\dim(I_Z) = k$.*

Proof. The case $k = \infty$ is trivial; any finite sequence Z will work. So we assume that k is a positive integer in the rest of the proof.

We first consider the case $p = \infty$ and $k > 1$. In this case, we consider $Z = \Lambda_\alpha - \{a_1, \dots, a_{k-1}\}$, where a_1, \dots, a_{k-1} are (any) distinct points in Λ_α , and

$$f(z) = \frac{\sigma_\alpha(z)}{(z - a_1) \cdots (z - a_{k-1})}.$$

It follows from Lemma 1 that $f \in F_\alpha^\infty$ and Z is exactly the zero sequence of f . Furthermore, if h is a polynomial of degree less than or equal to $k - 1$, then the function $f(z)h(z)$ is still in F_α^∞ .

On the other hand, if F is any function in F_α^∞ that vanishes on Z , then we can write

$$F(z) = f(z)g(z) = \frac{\sigma_\alpha(z)g(z)}{(z - a_1) \cdots (z - a_{k-1})},$$

where g is an entire function. For any positive integer n let C_n be the boundary of the square centered at 0 with horizontal and vertical side-length $(2n + 1)/\sqrt{\pi/\alpha}$. It is clear that

$$d(C_n, \Lambda_\alpha) \geq \sqrt{\pi/\alpha}/2, \quad n \geq 1.$$

So there exists a positive constant C such that

$$|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2} \geq C, \quad z \in C_n, n \geq 1.$$

This together with the assumption that $F \in F_\alpha^\infty$ implies that there exists another positive constant C such that

$$|g(z)| \leq C|z - a_1| \cdots |z - a_{k-1}| \quad (2)$$

for all $z \in C_n$ and $n \geq 1$. By Cauchy's integral estimates, the function g must be a polynomial of degree at most $k - 1$.

Therefore, when $p = \infty$, $k > 1$, and $Z = \Lambda_\alpha - \{a_1, \dots, a_{k-1}\}$, we have shown that a function $F \in F_\alpha^\infty$ vanishes on Z if and only if

$$F(z) = \frac{\sigma_\alpha(z)h(z)}{(z - a_1) \cdots (z - a_{k-1})},$$

where h is a polynomial of degree less than or equal to $k - 1$. This shows that $\dim(I_Z) = k$.

When $p = \infty$ and $k = 1$, we simply take $Z = \Lambda_\alpha$. The arguments above can be simplified to show that a function $F \in F_\alpha^\infty$ vanishes on Z if and only if $F = c\sigma_\alpha$ for some constant c .

Next, we assume that $0 < p < \infty$ and k is a positive integer. In this case, we let N denote the smallest positive integer such that $Np > 2$, or equivalently,

$$\int_{|z|>1} \left| \frac{\sigma_\alpha(z)e^{-\frac{\alpha}{2}|z|^2}}{z^N} \right|^p dA(z) < \infty. \quad (3)$$

Remove any $N + k - 1$ points $\{a_1, \dots, a_{N+k-1}\}$ from Λ_α and denote the remaining sequence by Z . Then Z is the zero sequence of the function

$$\frac{\sigma_\alpha(z)}{(z - a_1) \cdots (z - a_{N+k-1})},$$

which belongs to F_α^p in view of (3). In fact, if g is any polynomial of degree less than or equal to $k - 1$, then it follows from (3) that g times the above function belongs to I_Z .

Conversely, if f is any function in F_α^p that vanishes on Z , then we can write

$$f(z) = \frac{\sigma_\alpha(z)g(z)}{(z - a_1) \cdots (z - a_{N+k-1})},$$

where g is an entire function. Since $F_\alpha^p \subset F_\alpha^\infty$, it follows from (2) and Cauchy's integral estimates that g is a polynomial with degree less than or equal to $N + k - 1$. If the degree of g is $j > k - 1$, then

$$\frac{g(z)}{(z - a_1) \cdots (z - a_{N+k-1})} \sim \frac{1}{z^{N+k-1-j}}, \quad z \rightarrow \infty.$$

This together with $f \in F_\alpha^p$ shows that (3) still holds when N is replaced by $N + k - 1 - j$, which contradicts our minimality assumption on N . Thus $j \leq k - 1$, which shows that I_Z is k -dimensional. \square

Lemma 5. *Let Z be a zero sequence for F_α^p and $\dim(I_Z) = k < \infty$. Then $Z \cup \{a_1, \dots, a_k\}$ is always a uniqueness set for F_α^p .*

Proof. Let $Z' = Z \cup \{a_1, \dots, a_k\}$. If there exists a function $f \in F_\alpha^p$, not identically zero, such that f vanishes on Z' . Then the functions

$$f(z), \quad \frac{f(z)}{z - a_1}, \quad \dots, \quad \frac{f(z)}{z - a_k},$$

all belong to F_α^p and vanish on Z (obvious adjustments should be made when there are zeros of high multiplicity). They are clearly linearly independent, so the dimension of I_Z is at least $k + 1$. Therefore, the condition $\dim(I_Z) \leq k$ implies that $Z \cup \{a_1, \dots, a_k\}$ is always a uniqueness set for F_α^p . \square

Lemma 6. *Let Z be a zero sequence for F_α^p and $\dim(I_Z) > k$. Then $Z \cup \{a_1, \dots, a_k\}$ is never a uniqueness set for F_α^p .*

Proof. If $\dim(I_Z) > k$, there exist $k + 1$ linearly independent functions in I_k , say f_1, \dots, f_k, f_{k+1} . Fix any collection $\{a_1, \dots, a_k\}$ and let $Z' = Z \cup \{a_1, \dots, a_k\}$. Consider the linear combination

$$f = c_1 f_1 + \dots + c_{k+1} f_{k+1}, \quad (4)$$

and the system of linear equations

$$c_1 f(a_j) + \dots + c_{k+1} f_{k+1}(a_j) = 0, \quad 1 \leq j \leq k.$$

Again, obvious adjustments should be made when there are zeros of high multiplicity. This homogeneous system has k equations but $k + 1$ variables, so it always has nonzero solutions c_j , $1 \leq j \leq k$. With such a choice of c_j , the function f defined in (4) is not identically zero but vanishes on Z' . So Z' is not a uniqueness set for F_α^p . \square

Corollary 7. *Suppose Z is a zero sequence for F_α^p and k is a positive integer. Then the following conditions are equivalent.*

- (a) $\dim(I_Z) \leq k$.
- (b) $Z \cup \{a_1, \dots, a_k\}$ is a uniqueness set for F_α^p for all $\{a_1, \dots, a_k\}$.
- (c) $Z \cup \{a_1, \dots, a_k\}$ is a uniqueness set for F_α^p for some $\{a_1, \dots, a_k\}$.

Proof. That (a) implies (b) follows from Lemma 5 and the fact that expanding a uniqueness set always results in a uniqueness set. It is trivial that (b) implies (c). Lemma 6 shows that (c) implies (a). \square

Corollary 8. *Let Z be a zero sequence for F_α^p and k be a positive integer. Then the following conditions are equivalent.*

- (a) $\dim(I_Z) = k$.
- (b) For any $\{a_1, \dots, a_k\}$ the sequence $Z \cup \{a_1, \dots, a_{k-1}\}$ is not a uniqueness set for F_α^p but the sequence $Z \cup \{a_1, \dots, a_k\}$ is.

- (c) For some $\{a_1, \dots, a_{k-1}\}$ the sequence $Z \cup \{a_1, \dots, a_{k-1}\}$ is not a uniqueness set for F_α^p but $Z \cup \{b_1, \dots, b_k\}$ is a uniqueness set for some $\{b_1, \dots, b_k\}$.

Proof. This is a direct consequence of Corollary 7. \square

Therefore, if $\dim(I_Z) = k$ is a positive integer, then adding k points to Z always results in a uniqueness set, but adding less than k points never results in a uniqueness set. We now show that the second assertion can be improved.

Corollary 9. *Suppose Z is a zero sequence for F_α^p , $\dim(I_Z) = k$ is a positive integer, and $j < k$. Then $Z \cup \{a_1, \dots, a_j\}$ is always a zero sequence for F_α^p .*

Proof. By Corollary 8, $Z' = Z \cup \{a_1, \dots, a_j\}$ is not a uniqueness set for F_α^p . So there exists a function $f \in F_\alpha^p$, not identically zero, such that f vanishes on Z' . By Corollary 8 again, the number of additional zeros of f cannot exceed $k - j$. If these additional zeros a are divided out of f by the corresponding factors $z - a$, the resulting function is still in F_α^p and vanishes exactly on Z' . Thus Z' is a zero sequence for F_α^p . \square

Consequently, if Z is a zero sequence for F_α^p and $\dim(I_Z) = k$, then adding less than k points to Z will always result in a zero sequence for F_α^p again, but adding k points to Z will always result in a uniqueness set (which is certainly not a zero sequence). Once again, this shows how unstable the zero sequences of F_α^p are. The following result describes the structure of I_Z when it is finite dimensional.

Theorem 10. *Suppose Z is a zero sequence for F_α^p and $\dim(I_Z) = k$ is a positive integer. Then there exists a function $g \in I_Z$ such that $I_Z = gP_{k-1}$, where P_{k-1} is the set of all polynomials of degree less than or equal to $k - 1$.*

Proof. Let f be a function in I_Z , not identically zero. Then its zero sequence must be of the form $Z \cup \{a_1, \dots, a_j\}$, where $j \leq k - 1$. Otherwise, we can come up with a set $Z' = Z \cup \{a_1, \dots, a_k\}$ such that f vanishes on Z' , which is a contradiction to Lemma 5.

It follows from the previous paragraph that every nonzero function $f \in I_Z$ must have order 2 and type $\alpha/2$. Otherwise, multiplication of f by arbitrary polynomials will still produce functions in I_Z , and we get functions in I_Z that have more than k additional zeros.

Now fix a function $g \in I_Z$ that vanishes exactly on Z . For any $f \in I_Z$ we have the factorization $f = gPe^h$, where P is a polynomial with $\deg(P) \leq$

$k - 1$ and h is entire. Since both f and g are of order 2 and type $\alpha/2$, the function h must be constant. This shows that $I_Z \subset gP_{k-1}$. But

$$\dim(I_Z) = k = \dim(gP_{k-1}),$$

so we must actually have $I_Z = gP_{k-1}$. \square

Obviously, the function g in Theorem 10 is essentially unique. More specifically, any two such functions can only differ by a constant multiple. This essentially unique function is determined by requiring it to have *exactly* Z as its zero sequence.

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